PROTOTYPE EXAMPLE

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4 × 6 foot double-hung wood-framed window

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix* of the two products would be *most profitable*. Therefore, an OR team has been formed to study this question.

The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

> Determine what the *production rates* should be for the two products in order to *maximize their total profit*, subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the *production rate* is defined as the number of batches produced per week.) *Any* combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
2. Number of hours of production time used in each plant for each batch produced of each new product.
3. Profit per batch produced of each new product. (*Profit per batch produced* was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly *constant* regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this *profit per batch produced* times *the number of batches produced.*)

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production processes for the new products. By analyzing cost data from these same engineers and the marketing division, along with a pricing decision from the marketing division, the accounting department developed estimates for the third category.

Table 3.1 summarizes the data gathered.

The OR team immediately recognized that this was a linear programming problem of the classic **product mix** type, and the team next undertook the formulation of the corresponding mathematical model.

TABLE 3.1 Data for the Wyndor Glass Co. problem

	Production Time per Batch, Hours		
	Product		
Plant	1	2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

a) Formulate and solve the problem.

b) Consider the same set of feasible solutions and determine the set of optimal solutions if the objective is
b1) $Max\ Z = 5x_1 + x_2$; b2) $Max\ Z = 6x_1 + 4x_2$; b3) $Min\ Z = -x_1 + x_2$
b4) $Min\ Z = x_1 - x_2$; b5) $Max\ Z = x_1$; b6) $Max\ Z = 4x_2$;

c) Top management wants to know the consequences if a minimum of $50,000 of profit is required.

d) Solve the initial problem assuming that the capacity of Plants 2 and 3 is unlimited. Repeat b) with this new feasible region.

e) Consider that the total of 18 h.w. available in Plant 3 must be used. Keeping the remaining initial constraints, identify and explain what is the new feasible region and optimal solution.

f) Solve the initial problem considering that the amount of windows cannot be smaller than the quadruple of the amount of doors.

Planning an Advertising Campaign

PROTOTYPE 2

The Profit & Gambit Co. produces cleaning products for home use. This is a highly competitive market, and the company continually struggles to increase its small market share. Management has decided to undertake a major new advertising campaign that will focus on the following three key products:

- A spray prewash stain remover.
- A liquid laundry detergent.
- A powder laundry detergent.

This campaign will use both television and the print media. A commercial has been developed to run on national television that will feature the liquid detergent. The advertisement for the print media will promote all three products and will include cents-off coupons that consumers can use to purchase the products at reduced prices. The general goal is to increase the sales of each of these products (but especially the liquid detergent) over the next year by a significant percentage over the past year. Specifically, management has set the following goals for the campaign:

- Sales of the stain remover should increase by at least 3 percent.
- Sales of the liquid detergent should increase by at least 18 percent.
- Sales of the powder detergent should increase by at least 4 percent.

Table 2.2 shows the estimated increase in sales for each *unit* of advertising in the respective outlets.[4] (A *unit* is a standard block of advertising that Profit & Gambit commonly purchases, but other amounts also are allowed.) The reason for −1 percent for the powder detergent in the Television column is that the TV commercial featuring the new liquid detergent will take away some sales from the powder detergent. The bottom row of the table shows the cost per unit of advertising for each of the two outlets.

Management's objective is to determine how much to advertise in each medium to meet the sales goals at a minimum total cost. FORMULATE AND SOLVE GRAPHICALLY.

table 2.2

	Increase in Sales per Unit of Advertising		
Product	Television	Print Media	Minimum Required Increase
Stain remover	0%	1%	3%
Liquid detergent	3	2	18
Powder detergent	−1	4	4
Unit cost	$1 million	$2 million	

Exercises

1.

(1) **3.1-11.*** The Omega Manufacturing Company has discontinued the production of a certain unprofitable product line. This act created considerable excess production capacity. Management is considering devoting this excess capacity to one or more of three products: call them products 1, 2, and 3. The available capacity on the machines that might limit output is summarized in the following table: (PVa) :)

Machine Type	Available Time (Machine Hours per Week)
Milling machine	500
Lathe	350
Grinder	150

The number of machine hours required for each unit of the respective products is

Productivity coefficient (in machine hours per unit)

Machine Type	Product 1	Product 2	Product 3
Milling machine	9	3	5
Lathe	5	4	0
Grinder	3	0	2

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 units per week. The unit profit would be \$50, \$20, and \$25, respectively, on products 1, 2, and 3. The objective is to determine how much of each product Omega should produce to maximize profit.

(a) Formulate a linear programming model for this problem.

C **(b)** Use a computer to solve this model by the simplex method.

1. (2) (PVb))

PV 10)

12. A farm family owns 125 acres of land and has $40,000 in funds available for investment. Its members can produce a total of 3,500 person-hours worth of labor during the winter months (mid-September to mid-May) and 4,000 person-hours during the summer. If any of these person-hours are not needed, younger members of the family will use them to work on a neighboring farm for $5/hour during the winter months and $6/hour during the summer.

Cash income may be obtained from three crops and two types of livestock: dairy cows and laying hens. No investment funds are needed for the crops. However, each cow will require an investment outlay of $1,200, and each hen will cost $9.

Each cow will require 1.5 acres of land, 100 person-hours of work during the winter months, and another 50 person-hours during the summer. Each cow will produce a net annual cash income of $1,000 for the family. The corresponding figures for each hen are: no acreage, 0.6 person-hour during the winter, 0.3 more person-hour during the summer, and an annual net cash income of $5. The chicken house can accommodate a maximum of 3,000 hens, and the size of the barn limits the herd to a maximum of 32 cows.

Estimated person-hours and income per acre planted in each of the three crops are

	Soybeans	Corn	Oats
Winter person-hours	20	35	10
Summer person-hours	50	75	40
Net annual cash income ($)	600	900	450

The family wishes to determine how much acreage should be planted in each of the crops and how many cows and hens should be kept to maximize its net cash income. Formulate the linear programming model for this problem.

1. (3) **8.2-9.** The Energetic Company needs to make plans for the energy systems for a new building. (PVe))

The energy needs in the building fall into three categories: (1) electricity, (2) heating water, and (3) heating space in the building. The daily requirements for these three categories (all measured in the same units) are

Electricity	30 units
Water heating	20 units
Space heating	50 units

The three possible sources of energy to meet these needs are electricity, natural gas, and a solar heating unit that can be installed on the roof. The size of the roof limits the largest possible solar heater to 40 units, but there is no limit to the electricity and natural gas available. Electricity needs can be met only by purchasing electricity (at a cost of \$50 per unit). Both other energy needs can be met by any source or combination of sources. The unit costs are

	Electricity	Natural Gas	Solar Heater
Water heating	\$150	\$110	\$70
Space heating	\$140	\$100	\$90

The objective is to minimize the total cost of meeting the energy needs.

1. (4)

Controlling Air Pollution

(PV i))

The NORI & LEETS CO., one of the major producers of steel in its part of the world, is located in the city of Steeltown and is the only large employer there. Steeltown has grown and prospered along with the company, which now employs nearly 50,000 residents. Therefore, the attitude of the townspeople always has been, What's good for Nori & Leets is good for the town. However, this attitude is now changing; uncontrolled air pollution from the company's furnaces is ruining the appearance of the city and endangering the health of its residents.

A recent stockholders' revolt resulted in the election of a new enlightened board of directors for the company. These directors are determined to follow socially responsible policies, and they have been discussing with Steeltown city officials and citizens' groups what to do about the air pollution problem. Together they have worked out stringent air quality standards for the Steeltown airshed.

The three main types of pollutants in this airshed are particulate matter, sulfur oxides, and hydrocarbons. The new standards require that the company reduce its annual emission of these pollutants by the amounts shown in Table 3.12. The board of directors has instructed management to have the engineering staff determine how to achieve these reductions in the most economical way.

The steelworks has two primary sources of pollution, namely, the blast furnaces for making pig iron and the open-hearth furnaces for changing iron into steel. In both cases the engineers have decided that the most effective types of abatement methods are (1) increasing the height of the smokestacks,[6] (2) using filter devices (including gas traps) in the smokestacks, and (3) including cleaner, high-grade materials among the fuels for the furnaces. Each of these methods has a technological limit on how heavily it can be used (e.g., a maximum feasible increase in the height of the smokestacks), but there also is considerable flexibility for using the method at a fraction of its technological limit.

Table 3.13 shows how much emission (in millions of pounds per year) can be eliminated from each type of furnace by fully using any abatement method to its technological limit. For purposes of analysis, it is assumed that each method also can be used less fully to achieve any fraction of the emission-rate reductions shown in this table. Furthermore, the fractions can be different for blast furnaces and for open-hearth furnaces. For either type of furnace, the emission reduction achieved by each method is not substantially affected by whether the other methods also are used.

After these data were developed, it became clear that no single method by itself could achieve all the required reductions. On the other hand, combining all three methods at full capacity on both types of furnaces (which would be prohibitively expensive if the company's

TABLE 3.12 Clean air standards for the Nori & Leets Co.

Pollutant	Required Reduction in Annual Emission Rate (Million Pounds)
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

[6]Subsequent to this study, this particular abatement method has become a controversial one. Because its effect is to reduce ground-level pollution by spreading emissions over a greater distance, environmental groups contend that this creates more acid rain by keeping sulfur oxides in the air longer. Consequently, the U.S. Environmental Protection Agency adopted new rules in 1985 to remove incentives for using tall smokestacks.

1.
4 (cont)

TABLE 3.13 Reduction in emission rate (in millions of pounds per year) from the maximum feasible use of an abatement method for Nori & Leets Co.

	Taller Smokestacks		Filters		Better Fuels	
Pollutant	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

products are to remain competitively priced) is much more than adequate. Therefore, the engineers concluded that they would have to use some combination of the methods, perhaps with fractional capacities, based upon the relative costs. Furthermore, because of the differences between the blast and the open-hearth furnaces, the two types probably should not use the same combination.

An analysis was conducted to estimate the total annual cost that would be incurred by each abatement method. A method's annual cost includes increased operating and maintenance expenses as well as reduced revenue due to any loss in the efficiency of the production process caused by using the method. The other major cost is the *start-up cost* (the initial capital outlay) required to install the method. To make this one-time cost commensurable with the ongoing annual costs, the time value of money was used to calculate the annual expenditure (over the expected life of the method) that would be equivalent in value to this start-up cost.

This analysis led to the total annual cost estimates (in millions of dollars) given in Table 3.14 for using the methods at their full abatement capacities. It also was determined that the cost of a method being used at a lower level is roughly proportional to the fraction of the abatement capacity given in Table 3.13 that is achieved. Thus, for any given fraction achieved, the total annual cost would be roughly that fraction of the corresponding quantity in Table 3.14.

The stage now was set to develop the general framework of the company's plan for pollution abatement. This plan specifies which types of abatement methods will be used and at what fractions of their abatement capacities for (1) the blast furnaces and (2) the open-hearth furnaces. Because of the combinatorial nature of the problem of finding a plan that satisfies the requirements with the smallest possible cost, an OR team was formed to solve the problem. The team adopted a linear programming approach, formulating the model summarized next.

TABLE 3.14 Total annual cost from the maximum feasible use of an abatement method for Nori & Leets Co. ($ millions)

Abatement Method	Blast Furnaces	Open-Hearth Furnaces
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

1. (5) **Regional Planning** (PVj))

The SOUTHERN CONFEDERATION OF KIBBUTZIM is a group of three kibbutzim (communal farming communities) in Israel. Overall planning for this group is done in its Coordinating Technical Office. This office currently is planning agricultural production for the coming year.

The agricultural output of each kibbutz is limited by both the amount of available irrigable land and the quantity of water allocated for irrigation by the Water Commissioner (a national government official). These data are given in Table 3.8.

The crops suited for this region include sugar beets, cotton, and sorghum, and these are the three being considered for the upcoming season. These crops differ primarily in their expected net return per acre and their consumption of water. In addition, the Ministry of Agriculture has set a maximum quota for the total acreage that can be devoted to each of these crops by the Southern Confederation of Kibbutzim, as shown in Table 3.9.

TABLE 3.8 Resource data for the Southern Confederation of Kibbutzim

Kibbutz	Usable Land (Acres)	Water Allocation (Acre Feet)
1	400	600
2	600	800
3	300	375

TABLE 3.9 Crop data for the Southern Confederation of Kibbutzim

Crop	Maximum Quota (Acres)	Water Consumption (Acre Feet/Acre)	Net Return ($/Acre)
Sugar beets	600	3	1,000
Cotton	500	2	750
Sorghum	325	1	250

Because of the limited water available for irrigation, the Southern Confederation of Kibbutzim will not be able to use all its irrigable land for planting crops in the upcoming season. To ensure equity between the three kibbutzim, it has been agreed that every kibbutz will plant the same proportion of its available irrigable land. For example, if kibbutz 1 plants 200 of its available 400 acres, then kibbutz 2 must plant 300 of its 600 acres, while kibbutz 3 plants 150 acres of its 300 acres. However, any combination of the crops may be grown at any of the kibbutzim. The job facing the Coordinating Technical Office is to plan how many acres to devote to each crop at the respective kibbutzim while satisfying the given restrictions. The objective is to maximize the total net return to the Southern Confederation of Kibbutzim as a whole.

1.(6) (PV K))

Distributing Goods through a Distribution Network

The Problem. The DISTRIBUTION UNLIMITED CO. will be producing the same new product at two different factories, and then the product must be shipped to two warehouses, where either factory can supply either warehouse. The distribution network available for shipping this product is shown in Fig. 3.13, where F1 and F2 are the two factories, W1 and W2 are the two warehouses, and DC is a distribution center. The amounts to be shipped from F1 and F2 are shown to their left, and the amounts to be received at W1 and W2 are shown to their right. Each arrow represents a feasible shipping lane. Thus, F1 can ship directly to W1 and has three possible routes (F1 → DC → W2, F1 → F2 → DC → W2, and F1 → W1 → W2) for shipping to W2. Factory F2 has just one route to W2 (F2 → DC → W2) and one to W1 (F2 → DC → W2 → W1). The cost per unit shipped through each shipping lane is shown next to the arrow. Also shown next to F1 → F2 and DC → W2 are the maximum amounts that can be shipped through these lanes. The other lanes have sufficient shipping capacity to handle everything these factories can send.

The decision to be made concerns how much to ship through each shipping lane. The objective is to minimize the total shipping cost.

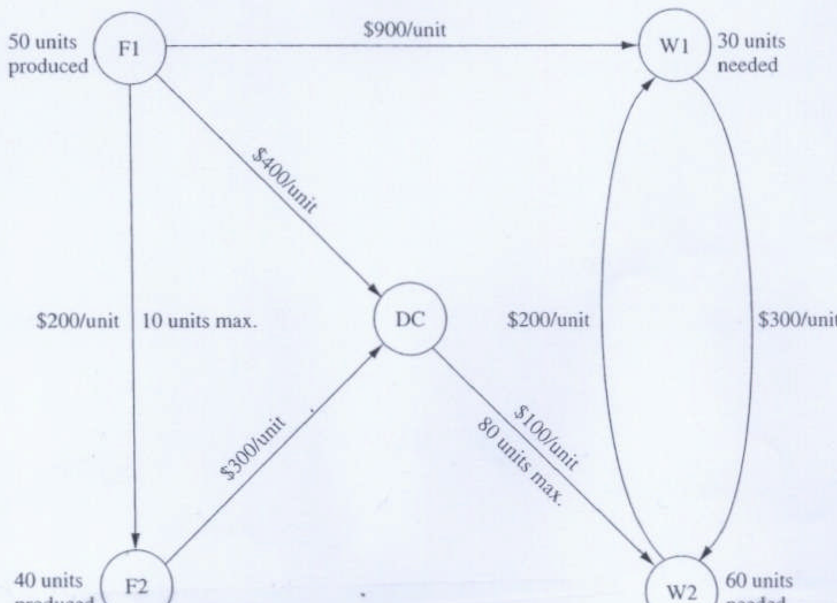

FIGURE 3.13
The distribution network for Distribution Unlimited Co.

2.

3.4-13.* Al Ferris has $60,000 that he wishes to invest now in order to use the accumulation for purchasing a retirement annuity in 5 years. After consulting with his financial adviser, he has been offered four types of fixed-income investments, which we will label as investments A, B, C, D.

Investments A and B are available at the beginning of each of the next 5 years (call them years 1 to 5). Each dollar invested in A at the beginning of a year returns $1.40 (a profit of $0.40) 2 years later (in time for immediate reinvestment). Each dollar invested in B at the beginning of a year returns $1.70 three years later.

Investments C and D will each be available at one time in the future. Each dollar invested in C at the beginning of year 2 returns $1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns $1.30 at the end of year 5.

Al wishes to know which investment plan maximizes the amount of money that can be accumulated by the beginning of year 6.

(a) All the functional constraints for this problem can be expressed as equality constraints. To do this, let A_t, B_t, C_t, and D_t be the amount invested in investment A, B, C, and D, respectively, at the beginning of year t for each t where the investment is available and will mature by the end of year 5. Also let R_t be the number of available dollars *not* invested at the beginning of year t (and so available for investment in a later year). Thus, the amount invested at the beginning of year t *plus* R_t must equal the number of dollars available for investment at that time. Write such an equation in terms of the relevant variables above for the beginning of each of the 5 years to obtain the five functional constraints for this problem.

(b) Formulate a complete linear programming model for this problem.

3. Use the graphical method and the *Solver/Excel* to solve the following LP problems:

a) $Max Z = x_1 + 2x_2$

s.t.:

$$\begin{cases} x_1 - 2x_2 \leq 3 \\ x_1 + x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases}$$

b) $Max Z = 3x_1 + 4x_2$

s. t.:

$$\begin{cases} x_1 - 2x_2 \geq 4 \\ x_1 + x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases}$$

c) $Max Z = x_1 + x_2$

s. t.:

$$\begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

d) $Max Z = x_1 - x_2$

s. t.:

$$\begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

e) $Max\ Z = -10x_1 - 5x_2$

s. t.:

$$\begin{cases} x_1 - x_2 \leq 5 \\ x_1 + \frac{8}{5} x_2 \geq -3 \\ x_1 \text{ livre; } x_2 \leq 0 \end{cases}$$

f) $Min Z = x_1 + x_2$

s. t.:

$$\begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

g) $Min Z = x_1 + x_2$

s. t.:

$$\begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

h) $Min Z = x_1 + x_2$

s. t.:

$$\begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq -2 \\ x_1, x_2 \geq 0 \end{cases}$$

i) $Min\ Z = 3x_1 + 2x_2$

s. t.:

$$\begin{cases} 2x_1 + x_2 \geq 8 \\ x_1 + 2x_2 \leq 12 \\ 2x_1 + 3x_2 = 12 \\ x_1, x_2 \geq 0 \end{cases}$$

j) $Max Z = 3x_1 + 6x_2$

s. t.:

$$\begin{cases} x_1 + 2x_2 \leq 4 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

4. Solve exercises 3.1-9., 3.1-10. and 3.2-2. from HL[1] (pág. 78-79).